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COMPUTING THE CONNES SPECTRUM OF A HOPF ALGEBRA

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ABSTRACT

Let H be a finite-dimensional Hopf algebra over the field K and let A be an H-module algebra. In a previous paper, we defined the Connes spectrum $\mathrm{CS}(A,H)$ for the action of H on A to be a certain subset of the set $\mathrm{Irr}(H)$ of irreducible representations of H. In this paper, we compute a number of examples; specifically, we consider certain inner and outer actions and we take a closer look at the cocommutative situation. We discover that the information encoded in the Connes spectrum is rather subtle and elusive.

1. Introduction

Let H be a finite-dimensional Hopf algebra over the field K and let A be an H-module algebra with 1. The Connes spectrum $\mathrm{CS}(A,H)$ for the action of H on A was defined in [OPQ] to be a certain subset of the set $\mathrm{Irr}(H)$ of irreducible representations of H. It was then shown, under suitable hypotheses, that $\mathrm{CS}(A,H) = \mathrm{Irr}(H)$ if and only if the smash product $A\#H$ is prime. In this paper, we continue to study the Connes spectrum; our goal here is to better understand its relationship to the H-action on A and we do this by computing a

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number of examples. Specifically, we consider certain inner and outer actions and we take a closer look at the cocommutative situation. The inner case is the most interesting; it indicates that the information encoded in the Connes spectrum is rather subtle and elusive.

We follow the notation of [OPQ]. Thus suppose that H is a finite-dimensional Hopf algebra over the field K and that A is an H-module algebra with 1. Then a hereditary subalgebra (without 1) of A is a subspace $B = RL$ where R is an H-stable right ideal of A and L is an H-stable left ideal. Note that B is necessarily H-stable, $B^2 \subseteq RAL = B$ and that $B = RL \subseteq R \cap L$. Furthermore, we let $\mathcal{H}(A,H)$ denote the set of all hereditary subalgebras $B \subseteq A$ with B reg B, that is with $\mathrm{l.ann}_B\, B = 0 = \mathrm{r.ann}_B\, B$.

Now let $\pi\colon H \to C$ be an irreducible representation of H and extend the left action of H on B to an action on $B \otimes_K C$ via the formula $h \cdot (b \otimes c) = (h \cdot b) \otimes c$ for all $b \in B$, $c \in C$. Let $X \in B \otimes C$; we define three subspaces of the tensor product as follows. First, $X \in B^m_\pi$ if and only if

$$\epsilon(h)X = \sum_{(h)} \Big[1 \otimes \pi(h_3)\Big](h_1 \cdot X)\Big[1 \otimes \pi\big(S^{-1}(h_2)\big)\Big] \tag{1.1}$$

for all $h \in H$. Next, $X \in B^l_\pi$ if and only if

$$\epsilon(h)X = \sum_{(h)} \Big[1 \otimes \pi(h_2)\Big](h_1 \cdot X) \tag{1.2}$$

for all $h \in H$. Finally, $X \in B^r_\pi$ if and only if

$$\epsilon(h)X = \sum_{(h)} (h_1 \cdot X)\Big[1 \otimes \pi\big(S^{-1}(h_2)\big)\Big] \tag{1.3}$$

for all $h \in H$. Of course, $\Delta h = \sum_{(h)} h_1 \otimes h_2$ is the comultiplication of h, the map $S\colon H \to H$ is the antipode and $\epsilon\colon H \to K$ is the counit of H. Furthermore, "m", "l" and "r" stand for "middle", "left" and "right", respectively. It was shown in [OPQ] that B^m_π is a subalgebra (without 1) of $B \otimes C$ and that $B^l_\pi B^r_\pi$ is a two-sided ideal of B^m_π.

With this notation, the Connes spectrum $\mathrm{CS}(A,H)$ is defined to be

$$\mathrm{CS}(A,H) = \{\, \pi \in \mathrm{Irr}(H) \mid B^l_\pi B^r_\pi \text{ reg } B^m_\pi \text{ for all } B \in \mathcal{H}(A,H) \,\}. \tag{1.4}$$

Note that, as given above, $\mathrm{CS}(A,H)$ makes sense even if H is not semisimple and K is not a splitting field for H. On the other hand, the latter conditions are certainly natural assumptions when dealing with $\mathrm{Irr}(H)$.

We close this section with some elementary observations. To start with, we have the following result which is reminiscent of [OPQ, Lemma 4.4]. Note that $A\#H$ is a free left and right A-module by [OPQ, Lemma 1.4(i)].

(1.5) LEMMA: *Let $B \in \mathcal{H}(A,H)$. If the smash product $A\#H$ is prime or semiprime, then so in $B\#H$.*

Proof: Suppose first that $A\#H$ is prime. Let I_1 and I_2 be ideals of $B\#H$ with $I_1I_2 = 0$ and let I_1' and I_2' be the possibly smaller ideals given by $I_i' = (B\#H)I_i(B\#H)$ for $i = 1,2$. Write $B = RL$ as a product of appropriate right and left ideals of A and set $J_i = LI_i'R \subseteq A\#H$. Since R and L are H-stable right and left ideals of A, respectively, and since $B\#H$ is closed under multiplication by H, it follows that each J_i is a two-sided ideal of $A\#H$. Furthermore, since $I_i' \subseteq I_i$, we have

$$J_1J_2 = LI_1'(RL)I_2'R \subseteq L(I_1'I_2')R = 0.$$

But $A\#H$ is prime, so this implies that $0 = J_j = LI_j'R$ for $j = 1$ or 2 and hence that $0 = RJ_jL = BI_j'B$. By assumption, B reg B and therefore, by freeness, B is regular in $B\#H$. Thus $0 = BI_j'B$ yields $0 = I_j' = (B\#H)I_j(B\#H)$ and, by regularity again, we deduce that $I_j = 0$. This handles the prime case; the semiprime result follows by taking $I_1 = I_2$. ∎

Recall that A is H-semiprime if A has no nonzero H-stable nilpotent ideal. In addition, H is said to be strongly semiprime if $A\#H$ is semiprime whenever A is an H-module algebra with 1 which is H-semiprime. It is clear that a finite-dimensional strongly semiprime Hopf algebra is necessarily semisimple.

(1.6) LEMMA: *Assume that $A\#H$ is semiprime.*

(i) *If $B \in \mathcal{H}(A,H)$ and if*

$$B^H = \{\, b \in B \mid \epsilon(h)b = h \cdot b \text{ for all } h \in H \,\}$$

is its subring of H-invariants, then B^H reg B.

(ii) *The counit ϵ is contained in $\mathrm{CS}(A,H)$.*

In particular, (i) and (ii) hold if H is strongly semiprime and A is an H-semiprime H-module algebra.

Proof:

(i) By the previous lemma, $B\#H$ is semiprime and this allows us to use the techniques of [BeM, Section 2]. Let f be a right integral of H and let $X = \mathrm{r.ann}_B B^H$. Since X is an H-stable right ideal of B, by [OPQ, Lemma 1.4(ii)], it follows that fX is a right ideal of $B\#H$. But $(fX)^2 = (fXf)X \subseteq fB^HX = 0$, so the semiprimeness of $B\#H$ implies that $fX = 0$ and hence that $X = 0$ by [OPQ, Lemma 1.4(i)]. In a similar manner, we can prove that $\mathrm{l.ann}_B B^H = 0$ and therefore we conclude that B^H reg B.

(ii) Note that ϵ: $H \to K$ is an irreducible representation of H and that $B \otimes K \cong B$ for any $B \in \mathcal{H}(A, H)$. Furthermore, equations (1.1), (1.2) and (1.3) easily imply that $B^m_\epsilon = B^l_\epsilon = B^r_\epsilon = B^H$. In view of the definition of $\mathrm{CS}(A, H)$, it suffices to show that B^H reg B^H and this follows from (i). ∎

It would be interesting to characterize those actions with $\mathrm{CS}(A, H) = \{\epsilon\}$. In particular, we would like to know whether this condition is equivalent to a natural property of $A\#H$.

2. Inner Actions

If H is a Hopf algebra over the field K, then H becomes an H-module algebra by way of the adjoint action

$$(\mathrm{ad}\, h)x = \sum_{(h)} h_1 x S(h_2)$$

for all $h, x \in H$, and it is clear that every two-sided ideal of H is $\mathrm{ad}\, H$-stable. In particular, if H is semisimple and if A is a simple two-sided ideal of H, then A is a K-algebra with 1 and A is an H-module algebra using the restriction of the adjoint representation. The goal of this section is to compute the Connes spectrum $\mathrm{CS}(A, H)$ in this situation.

We will actually start with certain weaker assumptions on H and A which will be described in the next paragraph. However, there is one natural assumption on the antipode S of H which will remain in force throughout the entire section. Namely, we suppose that the automorphism S^2 of H is inner, induced by the unit $u^{-1} \in H$. In other words,

$$S^2(h) = u^{-1} h u \quad \text{for all } h \in H. \tag{2.1}$$

Note that, if H is semisimple, then Kaplansky's conjecture asserts that $S^2 = 1$ and this conjecture has been verified for fields of characteristic 0 by [LR]. In particular, (2.1) holds in this case with $u = 1$. Furthermore, if K is a splitting field for H, then it is known [L] that S^2 is at least inner. In fact, even without the semisimple assumption, we have $S^2 = 1$ if H is cocommutative. In other words, (2.1) is not an unreasonable supposition; we conclude from it that $u^{-1}S^{-1}(h)u = S^2(S^{-1}(h)) = S(h)$ and therefore

$$\epsilon(h) = \sum_{(h)} h_1 u^{-1} S^{-1}(h_2) u = \sum_{(h)} u^{-1} S^{-1}(h_1) u h_2.$$

In particular, if we appropriately multiply each expression by u and u^{-1}, we obtain

$$\epsilon(h) = \sum_{(h)} u h_1 u^{-1} S^{-1}(h_2) = \sum_{(h)} S^{-1}(h_1) u h_2 u^{-1} \tag{2.2}$$

for all $h \in H$.

Now let H be an arbitrary finite-dimensional Hopf algebra satisfying (2.1), let A be a K-algebra with 1 and let $\theta\colon H \to A$ be a K-algebra homomorphism. Then θ and the adjoint action of H induce an action of H on A given by

$$h \cdot a = \sum_{(h)} \theta(h_1) a \theta(S(h_2)) \tag{2.3}$$

for all $h \in H$ and $a \in A$. In this way, A becomes an H-module algebra and we study the Connes spectrum $\mathrm{CS}(A, H)$ of this action. To start with, let $\pi\colon H \to C$ be an irreducible representation of H and recall that the action of H on $A \otimes C$ is given by

$$h \cdot (a \otimes c) = (h \cdot a) \otimes c = \sum_{(h)} \theta(h_1) a \theta(S(h_2)) \otimes c.$$

Thus, we have

$$h \cdot X = \sum_{(h)} \Big[\theta(h_1) \otimes 1\Big] X \Big[\theta(S(h_2)) \otimes 1\Big] \tag{2.4}$$

for all $h \in H$ and $X \in A \otimes C$.

(2.5) LEMMA: *Suppose B is a hereditary subalgebra of A and that $X \in B \otimes C \subseteq A \otimes C$.*

(i) $X \in B_\pi^m$ *if and only if*

$$\epsilon(h)X = \sum_{(h)} \Big[\theta(S^{-1}(h_4)) \otimes \pi(S^{-1}(h_1))\Big] X \Big[\theta(h_3) \otimes \pi(uh_2u^{-1})\Big]$$

for all $h \in H$.

(ii) $X \in B_\pi^l$ *if and only if*

$$\epsilon(h)X = \sum_{(h)} \Big[\theta(S^{-1}(h_3)) \otimes \pi(S^{-1}(h_1))\Big] X \Big[\theta(h_2) \otimes 1\Big]$$

for all $h \in H$.

(iii) $X \in B_\pi^r$ *if and only if*

$$\epsilon(h)X = \sum_{(h)} \Big[\theta(h_1) \otimes 1\Big] X \Big[\theta(S(h_2)) \otimes \pi(S^{-1}(h_3))\Big]$$

for all $h \in H$.

Proof:

(i) Equations (1.1) and (2.4) imply that $X \in B_\pi^m$ if and only if

$$\epsilon(h)X = \sum_{(h)} \Big[\theta(h_1) \otimes \pi(h_4)\Big] X \Big[\theta(S(h_2)) \otimes \pi(S^{-1}(h_3))\Big]$$

for all $h \in H$, and, since $S^{-1}\colon H \to H$ is onto, we can replace h by $S^{-1}(h)$ in the above expression. Hence, since

$$\Delta^3(S^{-1}(h)) = \sum_{(h)} S^{-1}(h_4) \otimes S^{-1}(h_3) \otimes S^{-1}(h_2) \otimes S^{-1}(h_1)$$

and $\epsilon(S^{-1}(h)) = \epsilon(h)$, we see that $X \in B_\pi^m$ if and only if

$$\epsilon(h)X = \sum_{(h)} \Big[\theta(S^{-1}(h_4)) \otimes \pi(S^{-1}(h_1))\Big] X \Big[\theta(h_3) \otimes \pi(S^{-2}(h_2))\Big].$$

But $S^{-2}(h) = uhu^{-1}$, so the result follows.

(ii) Here, equations (1.2) and (2.4) imply that $X \in B_\pi^l$ if and only if

$$\epsilon(h)X = \sum_{(h)} \Big[\theta(h_1) \otimes \pi(h_3)\Big] X \Big[\theta(S(h_2)) \otimes 1\Big]$$

for all $h \in H$. Again, replace h by $S^{-1}(h)$ and use

$$\Delta^2(S^{-1}(h)) = \sum_{(h)} S^{-1}(h_3) \otimes S^{-1}(h_2) \otimes S^{-1}(h_1).$$

Thus, since $\epsilon(S^{-1}(h)) = \epsilon(h)$, we see that $X \in B^l_\pi$ if and only if

$$\epsilon(h)X = \sum_{(h)} \Big[\theta(S^{-1}(h_3)) \otimes \pi(S^{-1}(h_1))\Big] X \Big[\theta(h_2) \otimes 1\Big]$$

as required.

(iii) This follows directly from equations (1.3) and (2.4). ∎

Now let us define a map D: $H \to A \otimes C$ by

$$D(h) = \sum_{(h)} \theta(h_2) \otimes \pi(uh_1u^{-1}). \tag{2.6}$$

Notice that D is the composite of the algebra homomorphisms

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{T} H \otimes H \xrightarrow{1\otimes u} H \otimes H \xrightarrow{\theta\otimes\pi} A \otimes C$$

where T is the twist map and $1 \otimes u$: $x \otimes y \mapsto x \otimes uyu^{-1}$. Thus D is also an algebra homomorphism. The following characterizations are a key ingredient in our computation. As in [OPQ], we use an underline to indicate the next expression to be simplified.

(2.7) LEMMA: *Let B be a hereditary subalgebra of A and let $X \in B \otimes C$.*

(i) *$X \in B^m_\pi$ if and only if*

$$D(h)X = XD(h) \quad \text{for all } h \in H.$$

(ii) *$X \in B^l_\pi$ if and only if*

$$D(h)X = X(\theta(h) \otimes 1) \quad \text{for all } h \in H.$$

(iii) *$X \in B^r_\pi$ if and only if*

$$XD(h) = (\theta(h) \otimes 1)X \quad \text{for all } h \in H.$$

Proof: We show that conditions (i), (ii) and (iii) as given above are equivalent to the corresponding conditions of Lemma 2.5.

(i) If $X \in B^m_\pi$, then for any $h \in H$ we have

$$\begin{aligned} D(h)X &= \Big[\sum_{(h)} \theta(h_2) \otimes \pi(u\underbrace{h_1}u^{-1})\Big]X \\ &= \Big[\sum_{(h)} \theta(h_3) \otimes \pi(uh_1u^{-1})\Big]\underbrace{\epsilon(h_2)X} \\ &= \sum_{(h)} \Big[\theta(\underbrace{h_6S^{-1}(h_5)}) \otimes \pi(\underbrace{uh_1u^{-1}S^{-1}(h_2)})\Big]X\Big[\theta(h_4) \otimes \pi(uh_3u^{-1})\Big] \end{aligned}$$

by Lemma 2.5(i). Thus (2.2) yields

$$\begin{aligned} D(h)X &= \sum_{(h)} \Big[\epsilon(h_4) \otimes \epsilon(h_1)\Big]X\Big[\theta(h_3) \otimes \pi(uh_2u^{-1})\Big] \\ &= \sum_{(h)} X\Big[\theta(\underbrace{h_3\epsilon(h_4)}) \otimes \pi(u\,\underbrace{\epsilon(h_1)h_2}\,u^{-1})\Big] \\ &= \sum_{(h)} X\Big[\theta(h_2) \otimes \pi(uh_1u^{-1})\Big] = XD(h). \end{aligned}$$

Conversely, suppose $D(h)X = XD(h)$ for all $h \in H$. Then, since $D(h_2) = \sum_{(h)} \theta(h_3) \otimes \pi(uh_2u^{-1})$, we have

$$\begin{aligned} &\sum_{(h)} \Big[\theta(S^{-1}(h_4)) \otimes \pi(S^{-1}(h_1))\Big]\underbrace{X\Big[\theta(h_3) \otimes \pi(uh_2u^{-1})\Big]} \\ &\quad = \sum_{(h)} \Big[\theta(\underbrace{S^{-1}(h_4)h_3}) \otimes \pi(\underbrace{S^{-1}(h_1)uh_2u^{-1}})\Big]X \\ &\quad = \sum_{(h)} \Big[\epsilon(h_2) \otimes \epsilon(h_1)\Big]X = \epsilon(h)X \end{aligned}$$

where (2.2) is used to simplify the expression $\sum_{(h)} S^{-1}(h_1)uh_2u^{-1}$. It therefore follows from Lemma 2.5(i) that $X \in B^m_\pi$.

(ii) If $X \in B^l_\pi$, then for any $h \in H$ we have

$$\begin{aligned} D(h)X &= \Big[\sum_{(h)} \theta(h_2) \otimes \pi(u\,\underbrace{h_1}\,u^{-1})\Big]X \\ &= \Big[\sum_{(h)} \theta(h_3) \otimes \pi(uh_1u^{-1})\Big]\underbrace{\epsilon(h_2)X} \\ &= \sum_{(h)} \Big[\theta(\underbrace{h_5S^{-1}(h_4)}) \otimes \pi(\underbrace{uh_1u^{-1}S^{-1}(h_2)})\Big]X\Big[\theta(h_3) \otimes 1\Big] \end{aligned}$$

by Lemma 2.5(ii). Thus, equation (2.2) yields

$$\begin{aligned} D(h)X &= \sum_{(h)} \Big[\epsilon(h_3)\otimes 1\Big] X \Big[\theta\big(\underbrace{\epsilon(h_1)h_2}\big)\otimes 1\Big] \\ &= \sum_{(h)} X\Big[\theta\big(\underbrace{h_1\epsilon(h_2}\big)\otimes 1\Big] = X\big(\theta(h)\otimes 1\big), \end{aligned}$$

as required.

Conversely, suppose that $D(h)X = X\big(\theta(h)\otimes 1\big)$ for all $h\in H$. Then we have

$$\begin{aligned} &\sum_{(h)} \Big[\theta(S^{-1}(h_3))\otimes\pi(S^{-1}(h_1))\Big] \underbrace{X\Big[\theta(h_2)\otimes 1\Big]} \\ &\quad= \sum_{(h)} \Big[\theta\big(\underbrace{S^{-1}(h_4)h_3}\big)\otimes\pi\big(\underbrace{S^{-1}(h_1)uh_2u^{-1}}\big)\Big]X \\ &\quad= \sum_{(h)} \Big[\epsilon(h_2)\otimes\epsilon(h_1)\Big]X = \epsilon(h)X \end{aligned}$$

by equation (2.2). Therefore $X\in B^l_\pi$, by Lemma 2.5(ii), and part (ii) is proved.

(iii) Finally, if $X\in B^r_\pi$, then Lemma 2.5(iii) implies that

$$\begin{aligned} XD(h) &= X\Big[\sum_{(h)}\theta(h_2)\otimes\pi(u\underbrace{h_1}u^{-1})\Big] \\ &= \sum_{(h)} \underbrace{\epsilon(h_1)X}\Big[\theta(h_3)\otimes\pi(uh_2u^{-1})\Big] \\ &= \sum_{(h)} \Big[\theta(h_1)\otimes 1\Big]X\Big[\theta(S(h_2)h_5)\otimes\pi\big(\underbrace{S^{-1}(h_3)uh_4u^{-1}}\big)\Big] \end{aligned}$$

for all $h\in H$. Thus, (2.2) yields

$$\begin{aligned} XD(h) &= \sum_{(h)} \Big[\theta(h_1)\otimes 1\Big]X\Big[\theta\big(S(h_2)\underbrace{\epsilon(h_3)h_4}\big)\otimes 1\Big] \\ &= \sum_{(h)} \Big[\theta(h_1)\otimes 1\Big]X\Big[\theta\big(\underbrace{S(h_2)h_3}\big)\otimes 1\Big] \\ &= \sum_{(h)} \Big[\theta\big(\underbrace{h_1\epsilon(h_2)}\big)\otimes 1\Big]X = (\theta(h)\otimes 1)X, \end{aligned}$$

as required.

On the other hand, suppose that $XD(h) = (\theta(h)\otimes 1)X$ for all $h \in H$. Then we have

$$\begin{aligned}\sum_{(h)} \underbrace{\left[\theta(h_1)\otimes 1\right]}X\left[\theta(S(h_2))\otimes \pi(S^{-1}(h_3))\right] &= \sum_{(h)} X\left[\theta(\underbrace{h_2S(h_3)})\otimes \pi(uh_1u^{-1}S^{-1}(h_4))\right]\\ &= \sum_{(h)} X\left[1\otimes \pi(uh_1u^{-1}S^{-1}(\underbrace{\epsilon(h_2)h_3}))\right]\\ &= \sum_{(h)} X\left[1\otimes \pi(\underbrace{uh_1u^{-1}S^{-1}(h_2)})\right] = \epsilon(h)X\end{aligned}$$

by equation (2.2), and therefore $X \in B_\pi^r$ by Lemma 2.5(iii). ∎

Next, we see that the hereditary subalgebras of A are easily determined in this context.

(2.8) LEMMA: *If $\theta\colon H \to A$ is an epimorphism, then the hereditary subalgebras of A are precisely its two-sided ideals.*

Proof: If I is a two-sided ideal of A, then equation (2.3) implies that I is H-stable. Thus $I = IA$ is a hereditary subalgebra of A.

For the converse, we need two elementary identities which follow from (2.3) and which hold for all $h \in H$ and $a \in A$. First,

$$\begin{aligned}\sum_{(h)} (\underbrace{h_1\cdot a})\theta(h_2) &= \sum_{(h)} \theta(h_1)a\theta(\underbrace{S(h_2)h_3})\\ &= \sum_{(h)} \theta(\underbrace{h_1\epsilon(h_2)})a = \theta(h)a\end{aligned}$$

and second,

$$\begin{aligned}\sum_{(h)} \theta(h_2)(\underbrace{S^{-1}(h_1)\cdot a}) &= \sum_{(h)} \theta(\underbrace{h_3S^{-1}(h_2)})a\theta(h_1)\\ &= \sum_{(h)} a\theta(\underbrace{h_1\epsilon(h_2)}) = a\theta(h).\end{aligned}$$

As a consequence of the former, we see that if R is an H-stable right ideal of A and if $a \in R$, then $\theta(h)a \in R$ for all $h \in H$. But $\theta(H) = A$, by assumption, and therefore R is a two-sided ideal of A. Similarly, the latter formula implies that any H-stable left ideal L of A is two sided. Hence, any $B = RL$ is a two-sided ideal of A. ∎

To proceed further, it is necessary to make some additional assumptions on A and on π and to introduce some additional notation. Once this is done, Lemma 2.7 can be given a module-theoretic interpretation, leading to a precise understanding of A_π^m, A_π^l and A_π^r. This, along with Lemma 2.8, will then yield the Connes spectrum.

To start with, let V be a fixed left H-module and assume that $A = \mathrm{End}_K(V)$ and that the homomorphism $\theta\colon H \to A = \mathrm{End}_K(V)$ is determined by the module action. Next, let $W = W(\pi)$ be the irreducible left H-module associated with the representation π and suppose that K is a splitting field for π. By this we mean that $C = \pi(H) = \mathrm{End}_K(W)$ and in particular that C is isomorphic to the full ring of $d_\pi \times d_\pi$ matrices over K where $d_\pi = \dim_K W$.

Since W is finite dimensional, it follows that

$$A \otimes C = \mathrm{End}_K(V) \otimes \mathrm{End}_K(W) = \mathrm{End}_K(V \otimes W)$$

with appropriate identification. In particular, any homomorphism from H to $A \otimes C = \mathrm{End}_K(V \otimes W)$ defines a left H-module structure on $V \otimes W$ and there are two such homomorphisms of interest to us. First, we have $D\colon H \to A \otimes C$, as given in (2.6), and we denote the corresponding H-module obtained in this way by $(V \otimes W)_D$. Next, we have $E\colon H \to A \otimes C$, given by

$$\text{(2.9)} \qquad E(h) = \theta(h) \otimes 1 \quad \text{for all } h \in H,$$

and we denote its corresponding left H-module by $(V \otimes W)_E$. In other words,

$$h(v \otimes w)_D = \sum_{(h)} \theta(h_2)v \otimes \pi(uh_1u^{-1})w$$

while $h(v \otimes w)_E = \theta(h)v \otimes w$.

(2.10) LEMMA: *With the above notation, we have*

(i) $(V \otimes W)_D \cong W \otimes V$, *where the latter is the usual tensor module given by*

$$h(w \otimes v) = \sum_{(h)} \pi(h_1)w \otimes \theta(h_2)v$$

for all $h \in H$, $w \in W$ *and* $v \in V$.

(ii) $(V\otimes W)_E \cong (\dim_K W)V$, *where the latter is the direct sum of* $\dim_K W$ *copies of* V.

Proof: For part (i), we observe that the map $W\otimes V \to (V\otimes W)_D$ given by $w\otimes v \mapsto v\otimes \pi(u)w$ is an H-module isomorphism. Part (ii) is obvious from the nature of the action of H on $(V\otimes W)_E$. ∎

Note that $(V\otimes W)_E \cong W_\epsilon \otimes V$ where $W_\epsilon = W$ as a K-vector space and where $hw = \epsilon(h)w$ for all $h\in H$ and $w\in W_\epsilon$. We are now ready to compute the sets A^m_π, A^l_π and A^r_π corresponding to the hereditary subalgebra $A \in \mathcal{H}(A,H)$.

(2.11) LEMMA: *With the above notation, we have*

(i) $A^m_\pi = \mathrm{End}_H\big((V\otimes W)_D\big)$

(ii) $A^l_\pi = \mathrm{Hom}_H\big((V\otimes W)_E,(V\otimes W)_D\big)$

(iii) $A^r_\pi = \mathrm{Hom}_H\big((V\otimes W)_D,(V\otimes W)_E\big)$

where these are all viewed as subspaces of $A\otimes C = \mathrm{End}_K(V\otimes W)$ *and where the endomorphisms act on the left.*

Proof: This is immediate from Lemma 2.7 and the definition of D, E and the corresponding modules $(V\otimes W)_D$ and $(V\otimes W)_E$. For example, the map X: $(V\otimes W)_E \to (V\otimes W)_D$ is an H-module homomorphism if and only if $X\in \mathrm{End}_K(V\otimes W) = A\otimes C$ and $XE(h) = D(h)X$ for all $h\in H$. In other words, by Lemma 2.7(ii), this occurs if and only if $X\in A^l_\pi$. The arguments for A^m_π and A^r_π are of course similar. ∎

As a consequence, we obtain

(2.12) LEMMA: *Suppose, in addition, that* V *is an irreducible* H*-module and that* H *is semisimple. Write* $(V\otimes W)_D = Y \dotplus Z$, *where* Y *is the homogeneous component corresponding to the irreducible module* V *and where* Z *is the sum of the remaining homogeneous components. Then*

$$A^m_\pi = \mathrm{End}_H\big((V\otimes W)_D\big) = \mathrm{End}_H(Y) \dotplus \mathrm{End}_H(Z)$$

is a ring direct sum and $A^l_\pi A^r_\pi = \mathrm{End}_H(Y)$.

Proof: Since H is semisimple, $(V\otimes W)_D$ does indeed have the structure $Y\dotplus Z$ as described above. Furthermore, since $\mathrm{Hom}_H(Y,Z)=0$ and $\mathrm{Hom}_H(Z,Y)=0$, it is clear that $A^m_\pi = \mathrm{End}_H\big((V\otimes W)_D\big)$ is the ring direct sum $\mathrm{End}_H(Y\dotplus Z) = \mathrm{End}_H(Y)\dotplus \mathrm{End}_H(Z)$. Finally, by Lemma 2.11(ii)(iii), $A^l_\pi A^r_\pi$ is the linear span of all H-endomorphisms of $(V \otimes W)_D$ which factor through $(V \otimes W)_E$. But $(V \otimes W)_E \cong (\dim_K W)V$, by Lemma 2.10(ii), and we know that Y is a direct sum of copies of V, so it is clear that $A^l_\pi A^r_\pi$ is indeed equal to $\mathrm{End}_H(Y)$. ∎

It is now a simple matter to prove the main result of this section.

(2.13) THEOREM: *Let H be a finite-dimensional semisimple Hopf algebra over the field K and assume that K is a splitting field for H. If V is an irreducible left H-module and if θ: $H \to A = \mathrm{End}_K(V)$ is its corresponding representation, then A becomes an H-module algebra via the action defined by*

$$h \cdot a = \sum_{(h)} \theta(h_1) a \theta(S(h_2)) \quad \text{for all } h \in H,\ a \in A.$$

In this situation, the Connes spectrum $\mathrm{CS}(A, H)$ is the set of all irreducible representations π of H with

$$W(\pi) \otimes V \cong d_\pi V$$

as H-modules. Here $W(\pi)$ is the irreducible module associated with π and $d_\pi = \dim_K W(\pi)$.

Proof: To start with, A is an H-module algebra with action satisfying (2.3). Furthermore, since H is semisimple and K is a splitting field of H, [L, Theorem 3.3] implies that S^2 is an inner automorphism of H and hence (2.1) holds. In other words, all the hypotheses of this section are satisfied. In particular, since θ: $H \to A$ is onto and since A is a simple ring, it follows from Lemma 2.8 that the only hereditary subalgebras of A are A itself and 0. Hence only $B = A$ need be considered when computing $\mathrm{CS}(A, H)$.

Let $\pi \in \mathrm{Irr}(H)$ and set $W = W(\pi)$. Since K is a splitting field for π, the previous lemma clearly implies that $A^l_\pi A^r_\pi$ reg A^m_π if and only if $(V \otimes W)_D = Y$ and hence if and only if $(V \otimes W)_D \cong dV$, a direct sum of d copies of V for some integer d. But $(V \otimes W)_D \cong W \otimes V$, by Lemma 2.10(i), so degree considerations imply that the isomorphism $(V \otimes W)_D \cong dV$ holds if and only if $W \otimes V \cong d_\pi V$ with $d_\pi = \dim_K W$. Thus, $\pi \in \mathrm{CS}(A, H)$ if and only if $W(\pi) \otimes V \cong d_\pi V$. ∎

In the context of the preceding theorem, we will frequently write $\mathrm{CS}(V)$ or $\mathrm{CS}(\theta)$ for the Connes spectrum $\mathrm{CS}(A, H)$.

3. Examples

Again, we assume throughout this section that H is a finite-dimensional semisimple Hopf algebra over K and that K is a splitting field for H. Our goal here is to look at specific examples related to Theorem 2.13. Recall that if $\theta\colon H \to \mathrm{End}_K(V)$ is a representation of H and if $V^* = \mathrm{Hom}_K(V, K)$ is the dual of V, then the contragredient representation $\theta^*\colon H \to \mathrm{End}_K(V^*)$ is defined by

$$(h\lambda)(v) = \lambda\big(S(h)v\big) \quad \text{for all } h \in H,\ \lambda \in V^* \text{ and } v \in V.$$

The following result is standard and quite elementary to prove. Note that a linear representation is a representation corresponding to an H-module of dimension 1.

(3.1) LEMMA: *Let H be a finite-dimensional semisimple Hopf algebra.*

(i) *If $\theta\colon H \to \mathrm{End}_K(V)$ is an irreducible representation of H, then so is $\theta^*\colon H \to \mathrm{End}_K(V^*)$. Furthermore, the map $V^* \otimes V \to K$ given by*

$$\lambda \otimes v \mapsto \lambda(v) \quad \textit{for all } \lambda \in V^*,\ v \in V$$

is an H-module epimorphism onto $K = W(\epsilon)$.

(ii) *The set of linear representations of H forms a group under $\otimes$. The identity element is the counit ϵ and the inverse of the representation π is its contragredient π^*.*

As a consequence, we have

(3.2) PROPOSITION: *Suppose $\theta\colon H \to \mathrm{End}_K(V)$ is an irreducible representation of H.*

(i) *If $\theta \neq \epsilon$, then $\theta^* \notin \mathrm{CS}(\theta)$.*

(ii) *If θ is linear, then $\mathrm{CS}(\theta) = \{\epsilon\}$.*

Proof:

(i) By Lemma 3.1(i), $V^* \otimes V$ has an irreducible constituent isomorphic to $W(\epsilon)$. Thus, since $V \not\cong W(\epsilon)$, Theorem 2.13 implies that θ^* is not contained in $\mathrm{CS}(\theta)$.

(ii) By Theorem 2.13, $\pi \in \mathrm{CS}(\theta)$ if and only if $d_\pi W(\theta) \cong W(\pi) \otimes W(\theta)$. Indeed, since θ is linear, Lemma 3.1(ii) implies that this occurs if and only if

$$d_\pi W(\theta) \otimes W(\theta^*) \cong W(\pi) \otimes W(\theta) \otimes W(\theta^*)$$

or equivalently

$$d_\pi W(\epsilon) \cong W(\pi) \otimes W(\epsilon) \cong W(\pi).$$

In other words, we must have $\pi = \epsilon$. ∎

In particular, if H is commutative, then all Connes spectra constructed in this manner just consist of the irreducible representation ϵ. This of course applies when $H = K[G]^*$ is the dual of a group algebra and also, by the result of [Ho], when $H = u(L)$ is a restricted enveloping algebra

Now let us assume that $H = K[G]$ is a group algebra. Here it is convenient to translate the results into the language of group characters. If π: $K[G] \to \mathrm{M}_{d_\pi}(K)$ is any representation of $K[G]$, let $\hat{\pi} : G \to K$ be its associated character. In other words, $\hat{\pi}(g) = \operatorname{tr} \pi(g)$ for all $g \in G$, where $\operatorname{tr} : \mathrm{M}_{d_\pi}(K) \to K$ is the usual matrix trace. Since $K[G]$ is semisimple and K is a splitting field, it is known that the character $\hat{\pi}$ uniquely determines the representation π. Furthermore, we know that the character of the tensor product $\theta \otimes \pi$ is just the product $\hat{\theta}\hat{\pi}$. If π is irreducible, then the kernel of $\hat{\pi}$ is defined by

$$\ker(\hat{\pi}) = \{\, g \in G \mid \hat{\pi}(g) = \hat{\pi}(1) = d_\pi \,\}.$$

It can be shown that $\ker(\hat{\pi})$ is the normal subgroup of G described by

$$\ker(\hat{\pi}) = \{\, g \in G \mid \pi(g) = \pi(1) \,\}$$

and, in particular, if $\pi \neq \epsilon$, then $\ker(\hat{\pi}) \neq G$.

(3.3) PROPOSITION: *If $H = K[G]$ and $\theta \in \mathrm{Irr}(H)$, then*

$$\mathrm{CS}(\theta) = \{\, \pi \in \mathrm{Irr}(H) \mid \hat{\theta}(g) = 0 \quad \text{for all } g \in G \setminus \ker(\hat{\pi}) \,\}.$$

In other words, $\pi \in \mathrm{CS}(\theta)$ if and only if $\hat{\theta}$ vanishes off $\ker(\hat{\pi})$.

Proof: By Theorem 2.13, $\pi \in \mathrm{CS}(\theta)$ if and only if $W(\pi) \otimes W(\theta) \cong d_\pi W(\theta)$. In terms of characters, this isomorphism occurs if and only if

$$\hat{\pi}(g)\hat{\theta}(g) = \hat{\pi}(1)\hat{\theta}(g) \quad \text{for all } g \in G$$

and the result follows immediately. ∎

We can now easily list a number of examples of interest. For this, we assume that the reader is reasonably familiar with group theory and character theory. Note that G is said to be an extraspecial p-group if $G' = \mathbb{Z}(G)$ has prime order p. Note also that part (v) below generalizes (i), but while part (i) is obvious, the proof of (v) requires that we quote a major theorem.

(3.4) Example: Let $H = K[G]$ and let $\theta \in \mathrm{Irr}(H)$.

(i) If G is a simple group, then $\mathrm{CS}(\theta) = \{\epsilon\}$ for all $\theta \in \mathrm{Irr}(H)$.

(ii) If G is an extrasprecial p-group and if θ is a nonlinear irreducible representation of $K[G]$, then $\mathrm{CS}(\theta)$ consists of all the linear representations of $K[G]$.

(iii) Suppose G has a unique nontrivial normal subgroup W and that G/W is cyclic of prime order p. If Ω denotes the set of linear representations ω of $K[G]$ with $\ker(\hat{\omega}) \supseteq W$, then $|\Omega| = p$ and

$$\mathrm{CS}(\theta) = \begin{cases} \Omega & \text{if } \theta \text{ restricted to } W \text{ is reducible, or} \\ \{\epsilon\} & \text{if } \theta \text{ restricted to } W \text{ is irreducible.} \end{cases}$$

(iv) $\mathrm{CS}(\theta)$ can contain representations of arbitrary degree.

(v) If $G/\mathbb{Z}(G)$ is simple, then $\mathrm{CS}(\theta) = \{\epsilon\}$ for all $\theta \in \mathrm{Irr}(H)$.

Proof: Let π and θ be irreducible representations of $K[G]$. We consider whether $\pi \in \mathrm{CS}(\theta)$. First, by Proposition 3.3, we know that $\epsilon \in \mathrm{CS}(\theta)$ since $\ker(\hat{\epsilon}) = G$. Now, set $N = \ker(\hat{\pi}) \triangleleft G$ and note that $N \neq G$ if $\pi \neq \epsilon$. Furthermore, if $G \neq 1$ and $N = 1$, then $\pi \notin \mathrm{CS}(\theta)$ since only multiples of the regular character can vanish off $N = 1$.

(i) The result is trivial for $G = 1$ and follows from the above comments for $G \neq 1$ since there are only two possibilities for N.

(ii) Here we know that $|G : \mathbb{Z}(G)| = p^{2n}$ for some integer $n \geq 1$, that $\theta(1) = p^n$ and that $\theta(g) = 0$ if and only if $g \notin \mathbb{Z}(G)$. Thus $\pi \in \mathrm{CS}(\theta)$ if and only if $\ker(\hat{\pi}) \supseteq \mathbb{Z}(G) = G'$ and hence if and only if π is linear.

(iii) In view of the comments of the first paragraph, we can assume that $N \neq 1$ and hence, by assumption, that $N \supseteq W$. In other words, $\pi \in \Omega$ and note that $\ker(\hat{\pi}) = W$ for all such $\pi \neq \epsilon$. From this we conclude that

$$\mathrm{CS}(\theta) = \begin{cases} \Omega & \text{if } \hat{\theta} \text{ vanishes off } W, \text{ or} \\ \{\epsilon\} & \text{if } \hat{\theta} \text{ does not vanish off } W. \end{cases}$$

Finally, if $\hat{\theta}$ vanishes off W, then the character inner product satisfies

$$[\hat{\theta}_W, \hat{\theta}_W]_W = |G/W|\,[\hat{\theta}, \hat{\theta}]_G = p > 1$$

and therefore θ_W, the restriction of θ to W, is reducible. On the other hand, if θ_W is assumed to be reducible, then since G/W is cyclic of prime order p, [I, Corollary 6.19] implies that $\theta_W = \phi_1 + \phi_2 + \cdots + \phi_p$ is a sum of p conjugate irreducible representations of $K[W]$. It then follows from Frobenius reciprocity that θ is a constituent of the induced representation ϕ_1^G and, by degree considerations, we have $\theta = \phi_1^G$. Thus $\hat{\theta}$ vanishes off W and this part is proved.

(iv) Let C be cyclic of prime order p, let W be an arbitrary finite group and set $G = C \wr W$, the wreath product of C by W. Choose K to be an algebraically closed field with $K[G]$ semisimple. Now G is the semidirect product of A by W, where A is the direct product of $w = |W|$ copies of C and where W acts on A by regularly permuting these factors. Say $A = \prod_1^w C_i$ and let λ be an irreducible representation of $K[A]$ with $\ker(\hat{\lambda}) = \prod_2^w C_i$. Then all W-conjugates of λ are distinct and hence the induced representation $\theta = \lambda^G$ of $K[G]$ is irreducible. Furthermore, $\hat{\theta}$ vanishes off A, so it follows that $\mathrm{CS}(\theta)$ contains any irreducible representation π with $\ker(\hat{\pi}) \supseteq A$. But $G/A \cong W$, so any irreducible representation of $K[W]$, lifted to $K[G]$, is contained in $\mathrm{CS}(\theta)$. Since W is arbitrary, we can find representations of arbitrary degree in suitable Connes spectra.

(v) Suppose $\pi \in \mathrm{CS}(\theta)$. Then $\hat{\theta}$ vanishes off N and, since $\hat{\theta}$ cannot vanish on any element of $\mathbb{Z}(G)$, it follows that $N \supseteq \mathbb{Z}(G)$. Thus, there are only two possibilities for N. If $N = \mathbb{Z}(G)$, then $\hat{\theta}$ vanishes off $\mathbb{Z}(G)$ and, by definition, this makes $\bar{G} = G/\mathbb{Z}(G)$ a group of central type. But central type groups are known to be solvable, by [HI], so this case cannot occur. Thus $N = G$ and $\pi = \epsilon$. ∎

Observe that (iii) above applies to the symmetric groups $G = \mathrm{Sym}_n$ with $n \geq 5$ and that (v) applies to the special linear groups $G = \mathrm{SL}_n(q)$ with $n \geq 2$ and with q a prime power. Of course, $q \geq 4$ when $n = 2$.

Again, suppose H is a finite-dimensional Hopf algebra over the field K and let A be an H-module algebra. If the action of H on A is purely inner, determined as in (2.3) by the homomorphism $\theta\colon H \to A$, then [BCM, Theorem 5.3] implies that the map $\tilde{}\colon H \to A\#H$ given by

$$h \mapsto \sum_{(h)} \theta(S(h_1))h_2 \quad \text{for all } h \in H$$

determines an algebra isomorphism between H and $\tilde{H} \subseteq \mathbf{C}_{A\#H}(A)$. Furthermore, it then follows that the smash product $A\#H$ is equal to the tensor product

$A \otimes \tilde{H}$. In particular, $A\#H$ is never prime when $\dim_K H > 1$. This is, of course, consistent with [OPQ, Theorem 1.6] and Proposition 3.2.

As will be apparent, more complicated smash products also exist in the context of inner actions, provided we allow θ to be a projective homomorphism. We will treat this topic quickly and in an elementary manner, without reverting to the study of twisted Hopf algebras.

To this end, suppose H and $\bar{H}$ are finite-dimensional semisimple Hopf algebras over the field K and let $\bar{\ }: H \to \bar{H}$ be a Hopf algebra epimorphism. If I is the kernel of $\bar{\ }$, then it is clear that $\epsilon(I) = 0$. Furthermore, $\bar{\ }$ determines a one-to-one correspondence between the irreducible representations of $\bar{H}$ and those irreducible representations of H with kernel containing I. Thus, with suitable identification, we can view $\mathrm{Irr}(\bar{H})$ as a subset of $\mathrm{Irr}(H)$. Now let A be an arbitrary H-module algebra and assume that I acts trivially on A so that $I \cdot A = 0$. We can then let $\bar{H}$ act on A via

$$(h + I) \cdot a = h \cdot a \quad \text{for all } h \in H,\ a \in A \tag{3.5}$$

and, in this way, A becomes an $\bar{H}$-module algebra.

(3.6) LEMMA: *With the above notation,*

$$\mathrm{CS}(A, \bar{H}) = \mathrm{CS}(A, H) \cap \mathrm{Irr}(\bar{H}).$$

Proof: Since $\bar{\ }: H \to \bar{H}$ is an epimorphism, it is clear that H and $\bar{H}$ have the same image in $\mathrm{End}_K(A)$ and therefore $\mathcal{H}(A, H) = \mathcal{H}(A, \bar{H})$. Next, if $\pi \in \mathrm{Irr}(\bar{H}) \subseteq \mathrm{Irr}(H)$, then it is clear that $C = \pi(H) = \pi(\bar{H})$. In other words, we check whether π is contained in $\mathrm{CS}(A, H)$ or in $\mathrm{CS}(A, \bar{H})$ by considering appropriate subsets of the same algebras $B \otimes C$ for all $B \in \mathcal{H}(A, H) = \mathcal{H}(A, \bar{H})$. But a close look at equations (1.1), (1.2), (1.3) and (3.5) shows that B^m_π, B^l_π and B^r_π are the same whether we view H or $\bar{H}$ as acting on $B \otimes C$. Thus $\pi \in \mathrm{CS}(A, H)$ if and only if $\pi \in \mathrm{CS}(A, \bar{H})$. ∎

With this, we can answer in the negative a question posed by J. Bergen and D. Haile. Again, we assume that the reader has a reasonable knowledge of finite group theory and character theory.

(3.7) Example: $\mathrm{CS}(A, H) = \{\epsilon\}$ need not imply that $A\#H$ is ring isomorphic to $A \otimes H$.

Proof: If $G = \mathrm{SL}_2(5)$, then $\mathbb{Z}(G) = \{1, z\}$ has order 2 and $G/\mathbb{Z}(G)$ is the simple group $\bar{G} = \mathrm{PSL}_2(5) \cong \mathrm{Alt}_5$. Let K be an algebraically closed field of characteristic 0 and set $H = K[G]$ and $\bar{H} = K[\bar{G}]$. Then the natural map $^{-}\colon H \to \bar{H}$ obtained from $G \to \bar{G}$ is certainly a Hopf algebra epimorphism. Note that the kernel of $^{-}$ is the ideal $I = (1-z)K[G]$.

Now $H = K[G]$ has an irreducible representation θ of degree 2 and thus, using (2.3), we can let $A = \mathrm{M}_2(K)$ be an H-module algebra. Next, since $\theta(z) \in K$, it follows that $1 - z$ acts trivially on A and hence that I acts trivially on A. Thus, equation (3.5) implies that A becomes an $\bar{H}$-module algebra via

$$(h + I)\cdot a = h \cdot a = \sum_{(h)} \theta(h_1) a \theta(S(h_2)) \quad \text{for all } h \in H,\ a \in A.$$

Since $\mathrm{CS}(A, H) = \{\epsilon\}$, by Example 3.4(v), we conclude from Lemma 3.6 that $\mathrm{CS}(A, \bar{H}) = \{\epsilon\}$. Our goal is to show that $A\#\bar{H}$ is not ring isomorphic to $A \otimes \bar{H}$.

Since $\bar{G}$ acts in an inner fashion of A, we can study the smash product $A\#\bar{H} = A\#K[\bar{G}]$ using standard techniques (see for example [P, Proposition 12.4 and Lemma 27.5]). Specifically, for each $x \in \bar{G}$, let u_x be a unit of A such that $\tilde{x} = u_x x$ centralizes A. Then we know that $E = \mathbb{C}_{A\#\bar{H}}(A)$ is equal to the twisted group algebra $K^t[\bar{G}]$ with group basis $\{\tilde{x} \mid x \in \bar{G}\}$. Furthermore, $A\#\bar{H} = A \otimes E$ and, if $T\colon A \to A$ denotes the matrix transpose, then the map $\rho\colon K^t[G] \to A$ given by

$$\rho\colon \sum_{x \in \bar{G}} k_x \tilde{x} \mapsto \Big(\sum_{x \in \bar{G}} k_x u_x\Big)^T$$

is an algebra homomorphism. Indeed, since $\theta\colon K[G] \to A$ is an epimorphism, it follows that $\{u_x \mid x \in \bar{G}\}$ spans A and hence that ρ is an epimorphism.

Now $E = K^t[\bar{G}]$ is semisimple and hence $E \cong \oplus \sum_i \mathrm{M}_{e_i}(K)$, a direct sum of suitable full matrix rings over K. Thus, since $A = \mathrm{M}_2(K)$, it follows that

$$A\#\bar{H} = A \otimes E \cong \oplus \sum_i \mathrm{M}_{2e_i}(K)$$

and, in particular, $A\#\bar{H}$ is semisimple. Suppose $A\#\bar{H}$ has a ring direct summand isomorphic to $\mathrm{M}_2(K)$. Then the uniqueness aspect of the Artin-Wedderburn Theorem implies that $e_i = 1$ for some i and hence $K^t[\bar{G}]$ has a linear representation λ. As is well known, this implies that $K^t[\bar{G}] \cong K[\bar{G}]$. Indeed, the proof of the latter isomorphism simply requires that we replace each $\tilde{x}$ as above by the unique element of $K\tilde{x}$ which maps to 1 under λ. It therefore follows that ρ gives rise to an algebra epimorphism from $K[\bar{G}]$ to $M_2(K)$ and hence $K[\bar{G}]$ must have an irreducible representation of degree 2. But this is a contradiction, since $K[\bar{G}] \cong K[\mathrm{Alt}_5]$ has irreducible representations of degree 1,3,3,4,5.

Thus we have shown that $A\#\bar{H}$ does not have a ring direct summand isomorphic to $M_2(K)$. On the other hand, $A \otimes \bar{H} = A \otimes K[\bar{G}]$ does have such a ring direct summand, since $K[\bar{G}]$ has the linear representation ϵ. In other words, $A\#\bar{H}$ is not ring isomorphic to $A \otimes \bar{H}$ and the result follows. ∎

Of course, Example 3.4 supplies numerous situations where $A\#H \cong A \otimes H$ does not imply that $\mathrm{CS}(A,H) = \{\epsilon\}$.

4. Outer Actions

We now move on to consider outer actions. Since there are a number of different, presumably inequivalent, definitions for this concept, we choose one which allows us to quickly compute the relevant Connes spectra. Thus suppose H is a finite-dimensional Hopf algebra over the field K and let A be an H-module algebra. As is well known, A is a left $A\#H$-module with action defined by

$$ah \bullet b = a(h \cdot b) \quad \text{for all } a, b \in A,\ h \in H$$

and A is a right $A\#H$-module with

$$b \bullet ha = \big(S^{-1}(h) \cdot b\big)a \quad \text{for all } a, b \in A,\ h \in H.$$

Furthermore, H is said to be trace outer on A if

$$(\alpha \bullet A)b \neq 0 \quad \text{and} \quad b(A \bullet \alpha) \neq 0 \tag{4.1}$$

for all $0 \neq \alpha \in A\#H$ and $0 \neq b \in A$.

The latter definition can be viewed in a more concrete manner as follows. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for H, let $0 \neq b \in A$ and let $c_1, c_2, \ldots, c_n$ be elements of A which are not all zero. Then $\alpha = \sum_i c_i x_i$ is a typical nonzero element of $A\#H$ and $\alpha \bullet a = \sum_i c_i(x_i \cdot a)$. Similarly, using [OPQ, Lemma 1.4(i)], $\alpha' = \sum_i S(x_i)c_i$ is a typical nonzero element of $A\#H$ and $a \bullet \alpha' = \sum_i (x_i \cdot a)c_i$. Thus, if we define the trace forms $\tau, \tau' \colon A \to A$ by

$$\begin{aligned} \tau(a) &= \sum_{i=1}^{n} c_i(x_i \cdot a)b \\ \tau'(a) &= \sum_{i=1}^{n} b(x_i \cdot a)c_i \end{aligned} \tag{4.2}$$

for all $a \in A$, then H is trace outer on A if and only if

$$(4.3) \qquad \tau(A) \neq 0 \quad \text{and} \quad \tau'(A) \neq 0 \qquad \text{for all such } \tau, \tau'.$$

Notice, for example, that if $H = K[G]$ is a group algebra, A is a prime ring and G is X-outer on A, then [P, Lemma 29.5(i)] implies that H is trace outer on A. Furthermore, if $H = K[G]^*$ is the dual of a group algebra, then $A = \sum_{g \in G} A_g$ is a G-graded ring and H is trace outer on A if and only if $aA_gb \neq 0$ for all $0 \neq a, b \in A$ and $g \in G$.

In the remainder of this section we assume that H is semisimple and we use e to denote the principal idempotent (integral) of H. Let B be a hereditary subalgebra of A, let $\pi \in \mathrm{Irr}(H)$ and set $C = \pi(H)$. Then we recall that H acts on $B \otimes_K C$ via the formula $h \cdot (b \otimes c) = (h \cdot b) \otimes c$ for all $b \in B$, $c \in C$ and $h \in H$.

(4.4) Lemma: *If $X \in B \otimes C$, then*

(i)
$$\sum_{(e)} \Big[1 \otimes \pi(e_2)\Big](e_1 \cdot X) \in B^l_\pi$$

(ii)
$$\sum_{(e)} (e_1 \cdot X)\Big[1 \otimes \pi\big(S^{-1}(e_2)\big)\Big] \in B^r_\pi.$$

Proof:

(i) Note that $B \otimes C$ becomes a left H-module when we define

$$hX = \sum_{(h)} \Big[1 \otimes \pi(h_2)\Big](h_1 \cdot X) \quad \text{for all } h \in H,\ X \in B \otimes C.$$

Furthermore, in view of equation (1.2), B^l_π is the set of H-invariants of this module. Thus $eX \in B^l_\pi$ for all $X \in B \otimes C$.

(ii) Similarly, $B \otimes C$ is a left H-module under the action

$$hX = \sum_{(h)} (h_1 \cdot X)\Big[1 \otimes \pi\big(S^{-1}(h_2)\big)\Big] \quad \text{for all } h \in H,\ X \in B \otimes C.$$

Moreover, in this case, B^r_π is the set of H-invariants by (1.3). Thus $eX \in B^r_\pi$ for all $X \in B \otimes C$ and the result follows. ∎

(4.5) LEMMA: *Write $\Delta(e) = \sum_{i=1}^{n} x_i \otimes y_i$ where $\{x_1, x_2, \ldots, x_n\}$ is a basis for H. Then the subsets of $A \otimes C$ given by*

(i) $\{1 \otimes \pi(y_1), 1 \otimes \pi(y_2), \ldots, 1 \otimes \pi(y_n)\}$, *and*

(ii) $\{1 \otimes \pi(S^{-1}(y_1)), 1 \otimes \pi(S^{-1}(y_2)), \ldots, 1 \otimes \pi(S^{-1}(y_n))\}$

are regular in $A \otimes C$, that is they annihilate no nonzero element of this algebra.

Proof: Since $\epsilon(e) = 1$, the identities for $\sum_{(e)} S(e_1)e_2$ and $\sum_{(e)} e_2 S^{-1}(e_1)$ yield

$$\sum_i S(x_i)y_i = 1 = \sum_i y_i S^{-1}(x_i).$$

Thus, since $1 \otimes \pi(1) = 1 \otimes 1$ annihilates no nonzero element of $A \otimes C$, the same is true of the set $\{1 \otimes \pi(y_1), 1 \otimes \pi(y_2), \ldots, 1 \otimes \pi(y_n)\}$. This proves (i) and, by applying S^{-1} to the above formulas, we obtain

$$\sum_i S^{-1}(y_i)x_i = 1 = \sum_i S^{-2}(x_i)S^{-1}(y_i)$$

and (ii) follows. ∎

Notice that any reasonable definition for an outer action should imply that $A\#H$ is prime and hence, under suitable hypotheses, that $\mathrm{CS}(A, H) = \mathrm{Irr}(H)$ by [OPQ, Theorem 1.6]. Thus the next result comes as no surprise.

(4.6) THEOREM: *Let H be a finite-dimensional strongly semiprime Hopf algebra over the field K and let A be an H-module algebra. Suppose that A is H-semiprime and that the action of H on A is trace outer. Then B_π^l and B_π^r are regular in $B \otimes \pi(H)$ for all $B \in \mathcal{H}(A, H)$ and $\pi \in \mathrm{Irr}(H)$. In particular, $\mathrm{CS}(A, H) = \mathrm{Irr}(H)$.*

Proof: Let $B \in \mathcal{H}(A, H)$ and $\pi \in \mathrm{Irr}(H)$ be given and set $C = \pi(H)$. We must show that B_π^l and B_π^r are regular in $B \otimes C$ and, for this, there are two left annihilators and two right annihilators which must be checked. Since the four arguments are essentially the same, we will prove in detail that $\mathrm{r.ann}_{B\otimes C} B_\pi^l = 0$ and then just briefly comment on the remaining cases. To start with, let $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be a K-basis for C. Then we know that every element of $B \otimes C$ is uniquely a sum of the form $\sum_i b_i \otimes \gamma_i$ with $b_i \in B$ and we call b_i the coefficient of γ_i.

Now let $Z \in \mathrm{r.ann}_{B\otimes C} B_\pi^l$ and assume by way of contradiction that $Z \neq 0$. If $\Delta(e) = \sum_1^n x_i \otimes y_i$ is written as in the preceding lemma, then Lemma 4.5(i) implies that $(1 \otimes \pi(y_j))Z \neq 0$ for some j, say $j = 1$. In particular, if we write $(1 \otimes \pi(y_1))Z \in \sum_i B \otimes \gamma_i$ in terms of the basis for C, then one of the γ_k coefficients is not zero. We can suppose that this occurs for γ_1 and that the coefficient is $0 \neq d \in B$. Now Lemma 1.6(i) implies that $B^H d \neq 0$ and hence we can choose $b \in B^H$ with $bd \neq 0$. Note that if $B = RL$ describes B as a product of a right and left ideal of A, then $BAB = R(LAR)L \subseteq RAL = B$ and therefore $bAb \subseteq B$.

Let $a \in A$ be arbitrary and set $X = bab \otimes 1 \in B \otimes C$ in Lemma 4.4(i). Then, since $b \in B^H$, it follows that B_π^l contains the element

$$\sum_i \Big[1 \otimes \pi(y_i)\Big]\Big[\big(x_i \cdot (bab)\big) \otimes 1\Big] = \sum_i b(x_i \cdot a)b \otimes \pi(y_i).$$

But $B_\pi^l Z = 0$, so this yields

$$0 = \sum_i \Big[b(x_i \cdot a)b \otimes 1\Big]\Big[1 \otimes \pi(y_i)\Big] Z$$

and, in particular, if $d_i \in B$ denotes the γ_1 coefficient of $(1 \otimes \pi(y_i))Z$, then

$$0 = \sum_i b(x_i \cdot a)bd_i.$$

Of course, the above holds for all $a \in A$ and hence corresponds to the vanishing of a trace form. Furthermore, we know that $b \neq 0$, that $bd_1 = bd \neq 0$ and that $\{x_1, x_2, \ldots, x_n\}$ is a basis for H. Thus the above trace form is nontrivial and this contradicts the fact that the action of H on A is trace outer. In other words, we must have $Z = 0$ and hence $\text{r.ann}_{B\otimes C} B_\pi^l = 0$. In a similar manner, we can show that $\text{l.ann}_{B\otimes C} B_\pi^l = 0$.

Finally, the proof that B_π^r is regular in $B \otimes C$ follows the same outline. Of course, here we must use parts (ii), rather than parts (i), of Lemmas 4.4 and 4.5. With these results in hand, we conclude that $B_\pi^l B_\pi^r$ reg $B \otimes C$ and hence that $B_\pi^l B_\pi^r$ reg B_π^m for all $B \in \mathcal{H}(A, H)$. In particular, $\pi \in \text{CS}(A, H)$ for all $\pi \in \text{Irr}(H)$. ∎

5. Cocommutative Algebras

In this final section, we consider certain special properties of the Connes spectrum machinery which arise when H is cocommutative. As usual, let H be a finite-dimensional Hopf algebra over the field K and let A be an H-module algebra with 1. Fix $\pi \in \text{Irr}(H)$, set $C = \pi(H)$ and recall that H acts on $A \otimes C$ via the formula $h \cdot (a \otimes c) = (h \cdot a) \otimes c$ for all $h \in H$, $a \in A$ and $c \in C$. We start with a general observation.

(5.1) LEMMA: *If $B \in \mathcal{H}(A, H)$, then B^l_π is a right $B^H \otimes C$-module and B^r_π is a left $B^H \otimes C$-module. Furthermore, $B^r_\pi B^l_\pi$ is a two-sided ideal of $B^H \otimes C$.*

Proof: Since $(B \otimes C)^H = B^H \otimes C$, the module properties follow immediately from equations (1.2) and (1.3). Now let $X \in B^r_\pi$ and $Y \in B^l_\pi$. Then for all $h \in H$, we have

$$\begin{aligned} h \cdot XY &= \sum_{(h)} (h_1 \cdot X)(\underbrace{h_2} \cdot Y) \\ &= \sum_{(h)} (h_1 \cdot X) \underbrace{\epsilon(h_3)}(h_2 \cdot Y) \\ &= \sum_{(h)} (h_1 \cdot X)\Big[1 \otimes \pi(S^{-1}(h_4))\Big] \underbrace{\Big[1 \otimes \pi(h_3)\Big](h_2 \cdot Y)} \\ &= \sum_{(h)} (\underbrace{h_1 \epsilon(h_2)} \cdot X)\Big[1 \otimes \pi(S^{-1}(h_3))\Big] Y \\ &= \sum_{(h)} \underbrace{(h_1 \cdot X)\Big[1 \otimes \pi(S^{-1}(h_2))\Big]} Y = \epsilon(h) XY \end{aligned}$$

by (1.2) and (1.3). Thus $XY \in (B \otimes C)^H = B^H \otimes C$. In other words, $B^r_\pi B^l_\pi \subseteq B^H \otimes C$ and the module properties yield the result. ∎

For the sake of simplicity, we will assume throughout the remainder of this section that H is cocommutative. This implies, among other things, that the antipode S of H is equal to its own inverse. Following [OPQ] and motivated by equation (1.1), we define the $*$ action of H on $A \otimes C$ by

$$h * X = \sum_{(h)} \Big[1 \otimes \pi(h_3)\Big](h_1 \cdot X)\Big[1 \otimes \pi(S^{-1}(h_2))\Big] \tag{5.2}$$

for all $h \in H$ and $X \in A \otimes C$. We will frequently write $*H$ for H to indicate that the action of H is given as in (5.2). Thus for example

(5.3) LEMMA: *Let H and A be as above.*

(i) *$A \otimes C$ is a $*H$-module algebra.*

(ii) *If $B \in \mathcal{H}(A, H)$, then $B^m_\pi = (B \otimes C)^{*H}$.*

(iii) *If I is a C-submodule of $A \otimes C$, then I is H-stable if and only if it is $*H$-stable.*

Proof:

(i) If $X = a \otimes c \in A \otimes C$, then

$$\begin{aligned} h * X &= (h_1 \cdot a) \otimes \Big[\pi(h_3) c \pi(S^{-1}(h_2))\Big] \\ &= (h_1 \cdot a) \otimes \Big[\pi(h_2) c \pi(S(h_3))\Big] \end{aligned}$$

by cocommutativity. Thus $*$ is the tensor action determined by $\cdot$ on A and by the adjoint composed with π on C. Since both A and C are H-module algebras under these actions, we use cocommutativity again to conclude that $A \otimes C$ is a H-module algebra under $*$. In other words, $A \otimes C$ is a $*H$-module algebra. This proves (i) and then part (ii) is immediate from equation (1.1).

(iii) Let $h \in H$ and $X \in A \otimes C$. Then cocommutativity yields

$$\begin{aligned} &\sum_{(h)} \Big[1 \otimes \pi(S(h_3))\Big] \underbrace{(h_1 * X)} \Big[1 \otimes \pi(h_2)\Big] \\ &\quad = \sum_{(h)} \Big[1 \otimes \pi(S(h_5) h_3)\Big] (h_1 \cdot X) \Big[1 \otimes \pi(S^{-1}(h_2) h_4)\Big] \\ &\quad = \sum_{(h)} \Big[1 \otimes \pi(\underbrace{S(h_4) h_5})\Big] (h_1 \cdot X) \Big[1 \otimes \pi(\underbrace{S^{-1}(h_2) h_3})\Big] \\ &\quad = \sum_{(h)} (\underbrace{h_1 \epsilon(h_2) \epsilon(h_3)} \cdot X) = h \cdot X. \end{aligned}$$

This formula, along with (5.1), now clearly yields the result. ∎

Next, we translate the definitions of B^l_π and B^r_π into the $*$ context. Part (ii) does not require that H is cocommutative.

(5.4) LEMMA: *Let $B \in \mathcal{H}(A, H)$ and let $X \in B \otimes C$.*

(i) *$X \in B^l_\pi$ if and only if*

$$\epsilon(h) X = \sum_{(h)} (h_1 * X) \Big[1 \otimes \pi(h_2)\Big] \quad \textit{for all } h \in H.$$

(ii) *$X \in B^r_\pi$ if and only if*

$$\epsilon(h) X = \sum_{(h)} \Big[1 \otimes \pi(S^{-1}(h_2))\Big] (h_1 * X) \quad \textit{for all } h \in H.$$

(iii) *B^l_π is a left B^m_π-module and B^r_π is a right B^m_π-module.*

Proof:

(i) If $h \in H$, then cocommutativity yields

$$\begin{aligned}\sum_{(h)} \underbrace{(h_1 * X)}\Big[1 \otimes \pi(h_2)\Big] &= \sum_{(h)} \Big[1 \otimes \pi(h_2)\Big](h_1 \cdot X)\Big[1 \otimes \pi\big(\underbrace{S(h_3)h_4}\big)\Big] \\ &= \sum_{(h)} \Big[1 \otimes \pi\big(\underbrace{h_2\epsilon(h_3)}\big)\Big](h_1 \cdot X) \\ &= \sum_{(h)} \Big[1 \otimes \pi(h_2)\Big](h_1 \cdot X).\end{aligned}$$

The new characterization of B^l_π now follows from equation (1.2).

(ii) Similarly, we have

$$\begin{aligned}\sum_{(h)} \Big[1 \otimes \pi\big(S^{-1}(h_2)\big)\Big]\underbrace{(h_1 * X)} &= \sum_{(h)} \Big[1 \otimes \pi\big(\underbrace{S^{-1}(h_4)h_3}\big)\Big](h_1 \cdot X)\Big[1 \otimes \pi\big(S^{-1}(h_2)\big)\Big] \\ &= \sum_{(h)} (h_1 \cdot X)\Big[1 \otimes \pi\big(S^{-1}(\underbrace{h_2\epsilon(h_3)})\big)\Big] \\ &= \sum_{(h)} (h_1 \cdot X)\Big[1 \otimes \pi\big(S^{-1}(h_2)\big)\Big].\end{aligned}$$

The new characterization of B^r_π is now clear from equation (1.3).

(iii) This follows from (i) and (ii) above since $B^m_\pi = (B \otimes C)^{*H}$. ∎

The subsets B^l_π and B^r_π need not be $*H$-stable. Nevertheless, we have

(5.5) LEMMA: *If $B \in \mathcal{H}(A, H)$, then $\mathrm{l.ann}_{B\otimes C} B^l_\pi$ and $\mathrm{r.ann}_{B\otimes C} B^r_\pi$ are both $*H$-stable.*

Proof: Let $X \in \mathrm{l.ann}_{B\otimes C} B^l_\pi$. Then part (i) of the previous lemma implies that, for any $h \in H$ and $Y \in B^l_\pi$, we have

$$\begin{aligned}(\underbrace{h} * X)Y &= \sum_{(h)} (h_1 * X)\underbrace{\epsilon(h_2)Y} \\ &= \sum_{(h)} \underbrace{(h_1 * X)(h_2 * Y)}\Big[1 \otimes \pi(h_3)\Big] \\ &= \sum_{(h)} (h_1 * \underbrace{XY})\Big[1 \otimes \pi(h_2)\Big] = 0\end{aligned}$$

since $*$ is a measuring and $XY = 0$. Thus $h * X \in \text{l.ann}_{B\otimes C} B^l_\pi$.

Now assume that $X \in \text{r.ann}_{B\otimes C} B^r_\pi$. If $h \in H$ and $Y \in B^r_\pi$, then part (ii) of the previous lemma yields

$$\begin{aligned} Y(\underbrace{h}*X) &= \sum_{(h)} \underbrace{\epsilon(h_2)Y}(h_1 * X) \\ &= \sum_{(h)} \Big[1 \otimes \pi(S^{-1}(h_3))\Big] \underbrace{(h_2 * Y)(h_1 * X)} \\ &= \sum_{(h)} \Big[1 \otimes \pi(S^{-1}(h_2))\Big] (h_1 * \underbrace{YX}) = 0 \end{aligned}$$

since $YX = 0$ and H is cocommutative. Thus $h * X \in \text{r.ann}_{B\otimes C} B^r_\pi$ and the lemma is proved. ∎

The next result would be slightly easier to prove if K was assumed to be a splitting field for π. In that case, $C = \mathrm{M}_{d_\pi}(K)$ is a full matrix ring over K and the ideals of $A \otimes C$ are all extended from those of A.

(5.6) Lemma: *Assume that H is strongly semiprime and that A is H-semiprime. If $B \in \mathcal{H}(A, H)$, then $B \otimes C$ is H-semiprime and $*H$-semiprime.*

Proof: We first consider $B = A$ and restrict our attention to the $\cdot$ action. Since H acts trivially on C, it follows that $(A\otimes C)\#H = (A\#H)\otimes C$ and observe that the latter algebra is a direct summand of $(A\#H) \otimes H$. Next, since H is strongly semiprime and A is H-semiprime, we note that $A\#H$ is semiprime and then that $(A\#H)\otimes H$ is semiprime. Thus $(A\#H)\otimes C = (A\otimes C)\#H$ is semiprime and we conclude that $A \otimes C$ is H-semiprime. In view of Lemma 5.3(iii), $A \otimes C$ is also $*H$-semiprime.

Now let $B = RL \in \mathcal{H}(A, H)$ and observe that $B\otimes C = (R\otimes C)(L\otimes C)$. Note also that $R\otimes C$ and $L\otimes C$ are H-stable and $*H$-stable right and left ideals of $A \otimes C$ by Lemma 5.3(iii). Furthermore, since B reg B, freeness implies that B reg $(B \otimes C)$ and hence that $(B \otimes C)$ reg $(B \otimes C)$. In other words, $B \otimes C$ is contained in both $\mathcal{H}(A \otimes C, H)$ and $\mathcal{H}(A \otimes C, *H)$. By [OPQ, Lemma 4.4] and the properties of $A\otimes C$, we conclude that $B\otimes C$ is both H-semiprime and $*H$-semiprime. ∎

We can now obtain the main result of this section.

(5.7) Theorem: *Let H be a finite-dimensional cocommutative Hopf algebra over the field K and let A be an H-module algebra with 1. Assume that H is* *strongly semiprime and that A is H-semiprime. If $B \in \mathcal{H}(A, H)$, $\pi \in \operatorname{Irr}(H)$ and $C = \pi(H)$, then the following are equivalent.*

(i) $B^l_\pi B^r_\pi$ reg B^m_π.

(ii) $B^l_\pi B^r_\pi$ reg $(B \otimes C)$.

(iii) $\text{l.ann}_{B^m_\pi} B^l_\pi = 0 = \text{r.ann}_{B^m_\pi} B^r_\pi$.

(iv) $\text{l.ann}_{B\otimes C} B^l_\pi = 0 = \text{r.ann}_{B\otimes C} B^r_\pi$.

Proof:

(i) $\Rightarrow$ (ii) Let I denote the right or left annihilator of $B^l_\pi B^r_\pi$ in $B \otimes C$. Since $B^l_\pi B^r_\pi \subseteq B^m_\pi = (B \otimes C)^{*H}$, it follows that I is a $*H$-stable left or right ideal by [OPQ, Lemma 1.4(ii)]. Now assumption (i) implies that $0 = I \cap B^m_\pi = I \cap (B \otimes C)^{*H}$. Thus, since $B \otimes C$ is $*H$-semiprime and H is strongly semiprime, [OPQ, Proposition 4.3(ii)] implies that $I = 0$.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (iv) Let $I = \text{l.ann}_{B\otimes C} B^l_\pi$, so that I is a $*H$-stable left ideal of $B \otimes C$ by Lemma 5.5. Now assumption (iii) implies that $0 = I \cap B^m_\pi = I \cap (B \otimes C)^{*H}$. Thus, as above, we conclude that $I = 0$. The argument for $J = \text{r.ann}_{B\otimes C} B^r_\pi$ is similar.

(iv) $\Rightarrow$ (i) Let $\alpha \in B^m_\pi$ with $B^l_\pi B^r_\pi \alpha = 0$ and note that $\alpha B^l_\pi \subseteq B^l_\pi$ by Lemma 5.4-(iii). Thus, by Lemma 5.1, $B^r_\pi \alpha B^l_\pi$ is a two-sided ideal of $B^H \otimes C$ and this ideal is nilpotent since $B^l_\pi B^r_\pi \alpha = 0$. On the other hand, $B \otimes C$ is H-semiprime, so [OPQ, Proposition 4.3(i)] implies that $(B \otimes C)^H = B^H \otimes C$ is semiprime. Thus we conclude that $B^r_\pi \alpha B^l_\pi = 0$ and, in particular, that $B^r_\pi \alpha \subseteq \text{l.ann}_{B\otimes C} B^l_\pi = 0$ by assumption (iv). Furthermore, this yields $\alpha \in \text{r.ann}_{B\otimes C} B^r_\pi = 0$, by (iv) again, and hence we have shown that $\text{r.ann}_{B^m_\pi} B^l_\pi B^r_\pi = 0$. Since the argument for the left annihilator is similar, the theorem is proved. ∎

Note that condition (i) must be considered when we compute the Connes spectrum $\text{CS}(A, H)$. On the other hand, we suspect that the equivalent condition (iii) will turn out to be much easier to deal with in general.

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